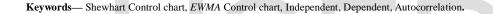# Studies on EWMA Control Chart in Presence of Autocorrelation

Ajit Goswami, Harendra Narayan Dutta

Abstract— Control charts are used to monitor a production process. While using control charts, a standard assumption is that observations from the process at different time periods are independent random variable. However, the assumption of independency in observation is not always true rather they are dependent in many cases. For example, in many chemical process industries, the observation taken at different time points are often correlated due to the dynamic nature of the process. Sometimes, some process inherently produced data that are autocorrelated. The presence of autocorrelation in process observations can have a large impact on traditional control chart developed under independent assumptions. When there is significant autocorrelation in the process observation, it is not advisable to apply traditional control chart methodology without modification, as there is typical effect of the autocorrelation on the control limits. In this paper, an attempt has been made to study the effect of autocorrelation on *EWMA* chart which is used alternative to Shewhart Control Chart specially in detecting small shifts in the process. Both for positive and negative autocorrelation, the effect on *EWMA* control limits are studied and in order to account for the autocorrelation, control limits are adjusted with the modification of *EWMA* control chart parameters.

Keywords— Shewhart Control chart, *EWMA* Control chart, Independent, Dependent, Autocorrelation.

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1. Introduction

A basic assumption in traditional application of Statistical Process Control (SPC) techniques is that the observations from the processes under investigation are normally and independently distributed. When these assumptions are satisfied, conventional control charts may be applied. However, the independence assumption is often violated in practice. The assumption of independent observations is not realistic for process of interest in many applications, because, the dynamics of the process produces autocorrelation in the process observation, such as in the chemical and pharmaceutical industries.

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Hence, the study of the effect of autocorrelation is an interesting one and it is very much relevant to the industry. In discrete as well as in continuous production process data often shows some autocorrelation, or serial dependence. Autocorrelation is present in the data generated by most continuous and batch process operations since the value of the particular parameter under monitoring is dependent on the previous value of that parameter.

Continuous product manufacturing operations such as the manufacture of food, chemicals, paper and other wood products often exhibits serial correlation. This phenomenon can also be present in monthly series of survey quality data. It is more apparent for data collected with frequent sampling but can also be due to the dynamics of the process. For instance, observations from automated test and inspection procedures where every quality characteristic is measured on every unit in time order of production, or

measurements of process variables from tanks, reactors and recycle streams in chemical processes are often highly correlated. Autocorrelation can also be evident in data arising from computer intrusion detection [Ye et al. [1] and Ye et al. [2]].

It is noteworthy to mention here that while dealing with autocorrelated data, so far there are two major approaches. These are:

(i)   One may adjust the control limits of existing SPC charts
(ii)   One may design residual based control charts.

In our present study, we will highlight some works on autocorrelated EWMA control charts based on Residual charts in section I. The study of effect of autocorrelation based on residual base EWMA chart has already been studied by the authors. Hence, in section II, the effect of autocorrelation on EWMA control limits are studied and in order to account for the autocorrelation, control limits are adjusted with the modification of *EWMA* control chart parameters both for positive as well as negative autocorrelation.

2. REVIEW OF PAST WORKS ON AUTO-CORRELATED EWMA CONTROL CHARTS

Beneke et.al. [3] investigated spectral control charts based on peridogram analysis and compared them with the Shewhart $\overline{X}$ and EWMA control charts. They recommended that the spectral control charts should be used along with existing control charts so that both shifts and cycles in the process mean can be detected.

Harvey and Fernandes [4] also Wisnowski and Keats [5] concluded that correlated count data can be modeled with a EWMA control chart.

Montgomery and Mastrangelo [6], emphasized that the use of EWMA is best for process observations that are positively autocorrelated at low lags and process mean that does not drift rapidly.

Harris and Ross [7] discussed the impact of autocorrelation on the performance of CUSUM

and EWMA charts, and showed that the average and median run lengths of these charts were sensitive to the presence of autocorrelation.

Wardell, Moskowitz and Plante [8] considered different control charts techniques in presence of data correlation and showed that the traditional EWMA chart is very good at detecting small shifts and perform well for large shifts when the autoregressive (AR) parameter is negative and Moving Average (MA) parameter is positive. Superville and Adamas [9] studied the performance analysis of CUSUM and EWMA control chart in detecting mean shift in AR processes. Tseng and Adams [10] generate in control ARLs with simulation for EWMA forecast errors. They compared the Shewhart, EWMA and CUSUM control chart. English [11] evaluated average run lengths of both $\overline{X}$ and EWMA charts by using analytical and simulation technique and tabulated for various process disturbance scenarios due to autocorrelation. Van Brackle and Reynolds [12] considered the EWMA and CUSUM control chart for the process mean when the observations are from an AR (1) process with additional random error. The numerical result of ARL shows that correlation can have a sufficient effect on the properties of these charts.

Schmid and Schone [13] proved theoretically that the run length of the auto correlated process is larger than in the case of independent variables provided that all the auto covariances are greater than or equal to zero.

Lu and Reynolds [14] considered the problem of monitoring the mean of a process in which the observation can be modeled as an AR (1) process plus a random error. A EWMA control chart based on the residual from the forecast values of the model is evaluated using an integral equation method.

Lu and Reynolds [15] extended his work on [14] to study the problem detecting special cause which may produce change in the process mean and variance. Several types of control charts and combinations of control charts are evaluated by them for their ability to detect changes in the process mean and variance. Jiang, et.al. [16]

proposed a new ARMA chart based on monitoring and ARMA statistics for the original observations. Lu and Reynolds [17] investigated CUSUM control charts for monitoring the process mean where observation from the process can be modeled as a AR(1) process with an added random error. A comparison of CUSUM charts to Shewhart and EWMA charts shows that CUSUM and EWMA charts were equally able to detect shift in the process mean.

Apley and Lee [18] gave a method for designing residual based EWMA charts under conditions of model uncertainty. The resulting EWMA control limits are modified to accommodate a number of variables including the level of model uncertainty.

Shiau and Chen [19] investigated the robustness of modified individual Shewhart control chart and modified exponentially weighted moving average (EWMA) control chart to the usual normality assumption of the white noise term in an AR(1) process with positive autocorrelation. They also found that the modified EWMA control chart is more robust to the normality assumption than the modified individuals Shewhart control chart in terms of the in-control ARL for some heavy-tailed symmetric distributions and some skewed distributions.

Patel and Divecha [20] introduces modified exponentially weighted moving average (modified EWMA) control chart which is very effective in detecting small and abrupt shifts in monitoring process mean.

Black et.al. [21] examines and compares the performance of Shewhart & EWMA control charts when the problem (the model) is subjected to autocorrelated with Weibull data. They also indicate that the EWMA chart outperforms the Shewhart in 62% of the cases, particularly those cases with low to moderate autocorrelation effects.

Suriyakat. W. et.al [22] considered the Fredholm second kind integral equations method to solve the corresponding Average Run Length (ARL), when the observations of a random process are serially-correlated.

3. EXPONENTIALLY WEIGHTED MOVING AVERAGE (EWMA) CONTROL CHARTS

The EWMA Control chart is a good alternative to Shewhart Control Chart when we are interested in detecting a small shift in a process. Though the performance of the EWMA Control chart is approximately equivalent to CUSUM charts, yet in some ways it is easier to execute. The EWMA Control chart was introduced by Roberts [23]. Later on, Crowder [24, 25], Lucas and Succucci [26] and Montgomery [27] studied extensively the various properties of EWMA Control chart. We however give the general theory of EWMA Control chart following Montgomery [27].

Let us consider,

$$W_t = \lambda \overline{X}_t + (1-\lambda)W_t \qquad \dots (1)$$

Where $0 < \lambda \le 1$ is a constant and the starting value (required with the first sample at i =1) is the process target, so that $W_0 = \mu$. The sequence of values W_t, $t = 0, 1, 2, \dots$ is called an exponentially weighted moving average. If we continually substitute for the W_t term on the right side of equation (1) we obtained that

$$W_t = \lambda \overline{X}_t + (1-\lambda)\left[\lambda \overline{X}_{t-1} + (1-\lambda)W_{t-2}\right]$$
$$= \lambda \overline{X}_t + \lambda(1-\lambda)\overline{X}_{t-1} + (1-\lambda)^2 W_{t-2}$$

$$= \lambda \overline{X}_t + \lambda(1-\lambda)\overline{X}_{t-1} + (1-\lambda)^2\left[\lambda \overline{X}_{t-2} + (1-\lambda)W_{t-3}\right]$$

$$= \lambda \overline{X}_t + \lambda(1-\lambda)\overline{X}_{t-1} + \lambda(1-\lambda)^2 \overline{X}_{t-2} + (1-\lambda)^3 W_{t-3}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$= \lambda \overline{X}_t + \lambda(1-\lambda)\overline{X}_{t-1} + \lambda(1-\lambda)^2 \overline{X}_{t-2} + \dots\dots + \lambda(1-\lambda)^{t-1}\overline{X}_t + (1-\lambda)^t \mu_0$$
$$\dots (2)$$

Thus we see from Equation (2) that W_t is a weighted average of all the subgroup averages up to time t, giving weight λ to the most recent subgroup and then successively decreasing the weight of earlier subgroup averages by the constant factor (1-λ), and then giving weight (1-λ)^t to the in control population mean.

The smaller the values of λ, the more even are the successive weights. For instance, if $\lambda = .1$ then the initial weight is .1 and the successive weights decrease by the factor .9; that is, the weight are .1, .09, .081, .073, .066, .059 and so on. On the other hand, if one choose, say $\lambda = .4$, then the successive weights are .4, .24, .144, .087, .052, ………. Since the successive weight $\lambda (1-\lambda)^{i-1}$, $i = 1, 2, 3, ………$, can be written as

$$\lambda(1-\lambda)^{i-1} = \overline{\lambda}e^{-\beta i}$$
Where
$$\overline{\lambda} = \frac{\lambda}{1-\lambda}, \qquad \beta = -\log(1-\lambda)$$

We say that the successive older data values are "exponentially weighted". To compute the mean and variance of the W_t, recall that, when in control, the subgroup averages $\overline{X}$, are independent normal random variable each having mean μ and variance $\frac{\sigma^2}{n}$. Therefore,

$$E[W_t] = \mu[\lambda + \lambda(1-\lambda) + \lambda(1+\lambda)^2 + …….. + \lambda(1-\lambda)^{t-1} + (1-\lambda)^t]$$

$$= \frac{\mu\lambda[1-(1-\lambda)^t]}{1-(1-\lambda)} + \mu(1-\lambda)^t$$
$$= \mu_0$$

To determine the variance, we again use the equation

$$Var(W_t) = \frac{\sigma^2}{n}\{\lambda^2 + [\lambda(1-\lambda)]^2 + [\lambda(1-\lambda)^3]^2 + …… + [\lambda(1-\lambda)^{t-1}]^2\}$$

$$= \frac{\sigma^2}{n}\lambda^2[1 + \beta + \beta^2 + ……. + \beta^{t-1}]$$

Where $\qquad \beta = (1-\lambda)^2$

$$= \frac{\sigma^2\lambda^2[1-(1-\lambda)^{2t}]}{n[1-(1-\lambda)^2]}$$

$$= \frac{\sigma^2\lambda[1-(1-\lambda)^{2t}]}{n(2-\lambda)}$$

Provided that the process has remained in control throughout, then for large values of t, the mean and variances reduced to:

$$E[W_t] = \mu_0 \qquad …(3)$$

$$Var(W_t) \approx \frac{\sigma^2\lambda}{n(2-\lambda)} \text{ since } \quad (1-\lambda)^{2t} \approx 0 \qquad …(4)$$

Thus, the upper and lower control limits for W_t are given by,

$$\left. \begin{array}{l} UCL = \mu_0 + 3\sigma\sqrt{\dfrac{\lambda}{n(2-\lambda)}} \\[3mm] LCL = \mu_0 - 3\sigma\sqrt{\dfrac{\lambda}{n(2-\lambda)}} \end{array} \right\} \qquad …(5)$$

4. MODEL DESCRIPTION:

LET US DEFINE THE PROCESS TO BE CONSIDERED BY

$$X_t = \mu + \xi_t \qquad …(6)$$

Where μ is constant and ξ_t satisfies

$$\xi_t = \rho_1\xi_{t-1} + \rho_2\xi_{t-2} + … + \rho_p\xi_{t-p} + \varepsilon_t$$
…(7)

Equation (7) is a p-th order autoregressive or AR (p) process.

If we now consider the autoregressive model of order 1, i.e., AR (1) model, which have many useful applications and simple as compared to higher order AR schemes, then we can expressed as

$$\xi_t = \rho_1\varepsilon_{t-1} + \varepsilon_t$$

Where $\quad -1 < \rho_1 < 1 \quad$ and ρ_1 is known as the coefficient of auto covariance and where ε_t is the stochastic disturbance term such that it satisfied the standard OLS assumptions, namely

$$E(\varepsilon_t) = 0$$
$$Var(\varepsilon_t) = \sigma_e^2$$
$$Cov(\varepsilon_t, \varepsilon_{t+s}) = 0; S \neq 0$$

In the *engineering literature*, an error term with the *preceding properties* is often called a **white noise error term** (Gujarati [28])

The process variance of model (6) is is given by

$$Var(X_t) = \frac{\sigma_\varepsilon^2}{(1-\rho_1^2)} \quad \dots (8)$$

Further, for large n, the sample $\overline{x}$ is the MLE of μ. Hence,

$$E(\overline{x_t}) = \mu$$

$$V(\overline{x_t}) \quad \text{is found out by Vasipoulos & }$$

Stamboulis [29] as:

$$V(\overline{x_t}) = \frac{\sigma^2}{n}\left[\frac{(1+\rho_1)}{(1-\rho_1)} - 2\left\{\frac{\rho_1(1-\rho_1^{\,n})}{n(1-\rho_1)^2}\right\}\right] \quad \dots (9)$$

$$= \frac{\sigma^2}{n}\phi(\rho_1, n)$$

Where

$$\phi(\rho_1, n) = \left[\frac{(1+\rho_1)}{(1-\rho_1)} - 2\left\{\frac{\rho_1(1-\rho_1^{\,n})}{n(1-\rho_1)^2}\right\}\right] \quad \dots (10)$$

Therefore, the modified EWMA control limits for the discussed model would be

$$UCL = \mu_0 + 3\sigma\sqrt{\frac{r}{n(2-\lambda)}\phi(\rho_1, n)}$$

$$CL = \mu_0 \quad \dots (11)$$

$$LCL = \mu_0 - 3\sigma\sqrt{\frac{r}{n(2-\lambda)}\phi(\rho_1, n)}$$

Where $\phi(\rho_1, n)$ is given in equation no. (10)

5. NUMERICAL ILLUSTRATIONS AND CONCLUSION

The specification of formalin product of a chemical factory (APL in Assam) is set up as 37± 0.5, by weight of Formaldehyde content. If the weight of formaldehyde gas in Formalin is below 36.5%, the customers do not accept it and if the same is above 37.5%, it is not affordable to manufacturer so far its cost-benefit margin are concerned. The data for thirty days has been collected by taking 5 samples from the recorded data sheet from the chemical factory.

To detect the small change that may have occurred during the production time of the given Formalin data we construct EWMA control limits choosing various values of λ and control limit L. The calculated control limits are shown in Table 1 below-

TABLE 1. CALCULATION OF UPPER AND LOWER CONTROL LIMITS

SET-I			
λ	L	$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}}$	$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}}$
0.05	2.6	37.3951	36.6049
0.06	2.6	37.4339	36.5661
0.07	2.6	37.4699	36.5301
0.08	**2.6**	**37.5037**	**36.4963**
0.09	2.6	37.5356	36.4644
0.10	2.6	37.5661	36.4339
SET-II			
λ	L	$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}}$	$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}}$
0.05	2.8	37.4255	36.5745
0.06	2.8	37.4673	36.5327
0.07	2.8	37.5061	36.4939
0.08	2.8	37.5424	36.4576
0.09	2.8	37.5738	36.4232
0.10	2.8	37.6096	36.3904

From the set of calculated control limits it is found to meet the APL factory specification, the following values of λ and L i.e. λ = 0.08 and L = 2.6 are suitable for constructing EWMA control limits which gives

UCL= 37+0.5037 = 37.50
CL = 37.00
LCL = 37-0.5037 = 36.50

Table 2 below gives the calculated EWMA control limits in presence of various values of positive as well as negative values of autocorrelation

Table 2. EWMA CONTROL LIMITS IN PRESENCE OF AUTOCORRELATION

ρ_1	λ	$\phi(\rho_1, n)$	L	UCL with autocorrelation	LCL with autocorrelation
0.1	0.8	1.17284	2.6	37.244	36.755
-0.1	0.8	0.8512	2.6	37.208	36.792
0.2	0.8	1.3750	2.6	37.264	36.735
-0.2	0.8	0.7222	2.6	37.191	36.808
0.3	0.8	1.6128	2.6	37.286	36.713
-0.3	0.8	0.6096	2.6	37.176	36.824
0.4	0.8	1.89344	2.6	37.310	36.689
-0.4	0.8	0.51104	2.6	37.161	36.838
0.5	0.8	2.225	2.6	37.336	36.663
-0.5	0.8	0.425	2.6	37.147	36.853

It is seen from the table that, the upper control limit has increases gradually as the positive autocorrelation coefficient increases and the upper control limits decreases as the negative autocorrelation coefficients decreases. Similarly,

the lower control limit decreases as the positive autocorrelation coefficient increases and the lower control limit increases as the negative autocorrelation coefficient increases. We may conclude that both the positive as well as negative autocorrelation have significant effect on the EWMA control limits.

To compensate the amount of autocorrelation, the user should adjust the control limits by choosing appropriate values of λ and L. For our above study, Table 3 shows the adjusted EWMA control limits for specific values of λ and L.

Table 3. ADJUSTED EWMA CONTROL LIMITS

ρ_1	λ	$\phi(\rho_1, n)$	L	UCL with autocorrelation	LCL with autocorrelation
0.1	0.30	1.17284	2.6	37.50	36.50
-0.1	0.35	0.8512	2.8	37.50	36.50
0.2	0.25	1.3750	2.7	37.50	36.50
-0.2	0.40	0.7222	2.75	37.50	36.50
0.3	0.20	1.6128	2.8	37.50	36.50
-0.3	0.50	0.6096	2.6	37.50	36.50
0.4	0.20	1.89344	2.6	37.50	36.50
-0.4	0.52	0.51104	2.8	37.50	36.50
0.5	0.15	2.225	2.8	37.50	36.50
-0.5	0.60	0.425	2.8	37.50	36.50

REFERENCES

[1] N. Ye, C. Borror and Y.B. Mang. "EWMA techniques for computer Intrusion detection through anomalous changes in event intensity", Quality and Reliability Engineering International, 2002, 18(6), 443-451.

[2] N. Ye, S. Vilbert and Q. Chen "Computer intrusion detection through EWMA for autocorrelated and uncorrelated data", IEEE transactions on Reliability, 2003, 52(1), 75-82.

[3] M.L. Beneke, R.E. Schiegel, and B.L. Foote "Spectral Analysis in Quality Control: A control Chart based on perodogram", Technometrics, 1988, Vol 30, 63-70.

[4] A.C. Harvey and C. Fernandes "Time Series Modeling for count or Correlated Observations", Jrnl. of Business and economic Statistics, 1989, 7(4), 407-422.

[5] J.W. Wisnowski and J.B. Keats "Monitoring the Availability of assets with Binomial and Correlated Observations", Quality Engineering, 1999, 11(3), 387-393.

[6] D.C. Montgomery and C.M. Mastrangelo "Some Statistical Process Control methods for Autocorrelated Data", JQT, 1991, 23(3), 179-204.

[7] T.J. Harris and W.H. Ross "Statistical Process control procedure for correlation observation", The Canadian Journal of chemical engineering, 1991, Vol. (69), 48-57.

[8] D.G. Wardell, H. Moskowitz and R.D. Plante "Control Chart in presence of Data Correlation", Management Science, 1992, Vol.38, 1084-1105.

[9] C.R. Superville, and B.M. Adams "An Estimation of Forecast-Based Quality Control Schemes", Commu. Stat. Simu-Compu., 1994, Vol23, 645-661.

[10] S. Tseng and B.M. Adams "Monitoring Autocorrelated Process with an Exponential Weighted Moving Average Forecast", Journal of Statistical Computation and Simulation., 1994, Vol. 50, 187-195.

[11] J.R. English "Fixed Control limits for continuous flow processes", IEEE transactions on Engineering Management, 1994, Vol-41, 310 -314.

[12] L.N. VanBrackle and M.R. Reynolds "EWMA and CUSUM Control Charts in the presence of Correlation", Commu. Statist-Simu-Compu, 1997, 26(4), 979-1008.

[13] W. Schmid and A. Schone "Some Properties of the EWMA Control Chart in the presence of Autocorrelation", Annals of Statistics, 1997, Vol.25, 1277-1283.

[14] C.W. Lu and M.R. Reynolds (Jr) "EWMA Control charts for monitoring the mean and Variance of Autocorrelated process", JQT, 1999a, Vol . 31(3), July, 259-274.

[15] C.W. Lu and M.R. Reynolds (Jr) "EWMA control charts for Monitoring the mean of Autocorrelated process", JQT, 1999b, Vol, 31(2), April, 166-188.

[16] W. Jiang, K.L. Tsui and W.H. Woodall "A new SPC monitoring method: the ARMA chart", Technometrices, 2000, Vol. 42(4), 399-410.

[17] C.W. Lu and M.R. Reynolds (Jr.) "CUSUM Chart for monitoring an Autocorrelated Process", JQT, 2001, Vol. 33(3), 316-334.

[18] D.W. Apley and H.C. Lee "Design of Exponentially Weighted Moving Average Control chairs for autocorrelated process with model uncertainty," Technometrics, 2003, Vol,-45, 187-198.

[19] J.J.H. Shiau & Y.H. Chen "Robustness of the EWMA control chart to Non-normality for Autocorrelated process", Quality Technology & Quantitative Management, 2005, Vol. 2, No. 2, Pp.125-146.

[20] A.K. Patel & J. Divecha "Modified Exponentially Weighted Moving Average (EWMA) control chart for an analytical process data", Journal of Chemical Engineering and Materials Science, 2011, Vol. 2(1), pp. 12-20.

[21] G. Black., J. Smith and S. Wells "The impact of Weibull data and autocorrelation on the performance of the Shewhart and exponentially weighted moving average control charts", International Journal of Industrial Engineering Computations 2, 2011, 575–582.

[22] W. Suriyakat. , Y. Areepong, S. Sukparungsee, and G. Mititelu "An Analytical Approach to EWMA Control Chart for Trend Stationary Exponential AR(1) Processes", Proceedings of the World Congress on Engineering, 2012, Vol I.

[23] S.W. Roberts "Control Chart Tests Based on Geometric Moving Average", Technometrics, 1959, vol.3, 239-250.

[24] S.V. Crowder "Computation of ARL for combined Individual Measurement and Moving Range Charts", Journal of Quality Technology, 1987b, Vol. 19, 161-164

[25] S.V. Crowder "Design of Exponentially Weighted Moving Average Scheme", Journal of Quality Technology, 1989, Vol. 21(3), 155-162.

[26] J.M. Lucas and M.S. Saccucci "Exponentially Weighted Moving Average Control Schemes: Properties and

Enhancement", Technometrics, 1990, Vol-32, 1-12.

[27]   D.C. Montgomery Introduction to Statistical Quality Control,
4th edition, John Wiley, 2001

[28]   D. Gujarati Basic Econometrics, 4th edition, McGraw-Hill,
2003

[29]   A.V. Vasilopoulos and A.P. Stamboulis "Modification of
Control chart Limits in the Presence of Data Correlation", JQT,
1978, 10(1), 20-30.

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